Theor. Comput. Fluid Dyn. (2010) 24:131–135
DOI 10.1007/s00162-009-0139-0

SPECIAL ISSUE ORIGINAL ARTICLE

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Explosive instability of geostrophic vortices. Part 2: parametric instability

Received: 17 October 2008 / Accepted: 24 June 2009 / Published online: 8 August 2009

Abstract In a two-layer quasi-geostrophic model, a baroclinic vortex is submitted to a periodic forcing of its mean baroclinic azimuthal velocity. It is shown that parametric effects could stabilize a vortex which is baroclinically unstable in the absence of forcing. Conversely, parametric resonance can destabilize a baroclinically stable vortex, under conditions on the vortex parameters, on the ratio of layer thicknesses or on the forcing frequency.

Keywords Quasi-geostrophic equations · Inviscid flows · Vortex stability · Parametric resonance

PACS 92.05.Bc, 92.10.ak, 92.10.ei, 47.15, 47.20.Cq

1 Introduction

In stratified rotating turbulence and in planetary fluids, vortices play an essential role in the transport of momentum, heat and tracers. The baroclinic instability of vortices has been studied with normal-mode perturbations [1–3]. The nonlinear evolution of these vortices, perturbed with normal modes, usually leads to multipolar vortices. Part 1 of this paper has compared the properties of linear baroclinic instability of vortices with piecewise-constant potential vorticity, perturbed with normal modes or with singular modes.

In a time-varying flow, the resonance of baroclinically neutral waves and of the forcing can lead to parametric instability, in particular for parallel flows [4]. Parametric instability has not yet been studied for vortex flows. Theoretical elements underlying this instability have been developed in [5]. Here, we consider the parametric resonance of neutral waves with a periodic external forcing which modifies the baroclinic velocity of this vortex. We study the properties of this parametric instability.

Communicated by H. Aref

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2 Model equations and parametric instability of an oscillating baroclinic vortex

The forced two-layer quasi-geostrophic equations describe the evolution of layerwise potential vorticity

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = E_j(r,t), \quad q_j = \nabla^2\psi_j + F_j(\psi_k - \psi_j)$$

where q_j is the layerwise potential vorticity (the subscripts $j = 1, 2$ denote upper and lower layers respectively, and $k = 3 - j$), F_j are the layer coupling coefficients ($F_j = f_0^2/g'H_j$), H_j is layer thickness, and $H = H_1 + H_2$. The internal deformation radius is $R_\mathrm{d} = \sqrt{g'H_1H_2}/f_0\sqrt{H}$ and γ is its inverse. The term $E_j(r,t)$ is a time periodic forcing of the baroclinic mean flow.

If neutral waves traveling on the vortex periphery resonate with this forcing (like a pendulum whose axis of rotation oscillates vertically), they can be amplified via the process of parametric instability.

Part 1 of the paper has shown that the linear instability of a two-layer vortex with piecewise-constant potential vorticity can be described in vertical modes by the equation $\partial_t X = AX$ with $X(\eta_t, \eta_c)$ and

$$\left[A = -i\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$$

Calling $V_t(1)$, $V_c(1)$ the barotropic and baroclinic velocities of the mean flow, at the vortex boundary, we have $a = lV_t(1)(1 - 1/l)$, $b = lV_c(1)(1 - 1/(2lI_1(\gamma)K_1(\gamma))$, $c = lV_c(1)(1 - (I_l(\gamma)K_l(\gamma))/(I_1(\gamma)K_1(\gamma))$ and $d = lV_t(1)(1 - 2I_l(\gamma)K_l(\gamma)) + \xi lV_c(1)(1 - (I_l(\gamma)K_l(\gamma))/(I_1(\gamma)K_1(\gamma))$, where $\xi = (1-\delta)/\sqrt{\delta}$ with $\delta = H_1/H_2$.

3 Parametric instability near marginality of baroclinic instability

We assume that the mean baroclinic velocity is close to that at marginality of baroclinic instability (called V_c^0 at the vortex boundary). The time periodic forcing adds a weak unsteady component to this velocity to allow parametric instability. The baroclinic azimuthal velocity at the vortex boundary is then written

$$V_c(1) = V_c^0[1 + \varepsilon h(t)]$$

with $\varepsilon \ll 1$, $h(t)$ is a time periodic function. Under these conditions, the linearized dynamics matrix is

$$A = A_0 + \varepsilon h(t) A_1$$

A_0 is given by the expression of A hereabout with $V_c(1) = V_c^0$. Note that marginality of baroclinic instability is defined by $(a_0 - d_0)^2 + 4b_0c_0 = 0$. The four terms of A_1 are given by $a_1 = 0$, $b_1 = V_c^0(1 - 1/(2lI_1(\gamma)K_1(\gamma))$, $c_1 = V_c^0(1 - (I_l(\gamma)K_l(\gamma))/(I_1(\gamma)K_1(\gamma))$ and $d_1 = \xi V_c^0(1 - (I_l(\gamma)K_l(\gamma))/(I_1(\gamma)K_1(\gamma))$. It is easily shown that matrices A_0 and A_1 do not commute, if $V_t \neq 0$, and therefore parametric instability is possible [5].

The calculations developed in [4] are adapted to our case. The contour perturbation X is expanded in powers of ε as $X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots$. The fast time is $t_0 = lb_0t$, slow times are defined by $t_1 = \varepsilon t_0$, $t_2 = \varepsilon^2 t_0 \ldots$.

First, we assume that $h(t) = H\cos(\omega t_0) + \varepsilon G$. H is the amplitude of the oscillatory part of the mean flow, G is the supercriticality of the mean flow (if positive, or the subcriticality if negative) and ω is the pulsation of the oscillatory mean flow. This will allow parametric resonance of neutral Rossby waves, associated with the potential vorticity jump at the vortex boundary, with the oscillatory baroclinic mean flow.

To simplify the linear instability equations, we define the new variables

$$Y_j = X_j\, exp(il(a_0 + d_0)t_0/2), \quad B_0 = A_0 + \frac{il}{2}(a_0 + d_0)Id,$$

where Id is the identity matrix. The linear equations are now written

$$\begin{aligned}
\partial_{t_0} Y_0 &= B_0 Y_0 \\
\partial_{t_0} Y_1 + \partial_{t_1} Y_0 &= B_0 Y_1 + H\cos(\omega t_0) A_1 Y_0 \\
\partial_{t_0} Y_2 + \partial_{t_1} Y_1 + \partial_{t_2} Y_0 &= B_0 Y_2 + H\cos(\omega t_0) A_1 Y_1 + G A_1 Y_0
\end{aligned}$$

Calling (y_{jt}, y_{jc}) the components of Y_j, and setting $\alpha = (d_0 - a_0)/(2b_0)$, $\beta = d_1/b_0$, the equations at zeroth order in ε are

$$\partial_{t_0} y_{0t} = i\alpha y_{0t} - i y_{0c}$$
$$\partial_{t_0} y_{0c} = -i\alpha y_{0t} + i\alpha^2 y_{0c}$$

By differentiating the equations in t_0, we can show that the neutral waves are described by $\partial_{t_0} Y_0 = 0$ and by $y_{0c} = \alpha y_{0t}$.

At first order, the equations are

$$\partial_{t_0} y_{1t} + \partial_{t_1} y_{0t} = i\alpha y_{1t} - i y_{1c} - i H \cos(\omega t_0) y_{0c}$$
$$\partial_{t_0} y_{1c} + \partial_{t_1} y_{0c} = -i\alpha y_{1t} + i\alpha^2 y_{1c} + i\alpha^2 H \cos(\omega t_0) y_{0t} - i\alpha\beta H \cos(\omega t_0) y_{0t}$$

We differentiate each equation in t_0 to find $\partial^2_{t_0} Y_1$, we make use of the fact that $\partial_{t_0} Y_0 = 0$, and then we integrate $\partial^2_{t_0} y_{1t}$ to find

$$y_{1t} = -\frac{H}{\omega^2} \left[\alpha(2\alpha - \beta)\cos(\omega t_0) + i\alpha\omega \sin(\omega t_0)\right] y_{0t}$$

and y_{1c} is readily obtained via the first equation of this first order system, as

$$y_{1c} = i\partial_{t_1} y_{0t} + \frac{iH}{\omega^2} \left[\omega\alpha(\alpha - \beta)\sin(\omega t_0) + i\alpha^2(2\alpha - \beta)\cos(\omega t_0)\right] y_{0t}$$

Finally, at second order in ε, the equations are

$$\partial_{t_0} y_{2t} + \partial_{t_1} y_{1t} + \partial_{t_2} y_{0t} = i\alpha y_{2t} - i y_{2c} - i H \cos(\omega t_0) y_{1c} - i G y_{0c}$$
$$\partial_{t_0} y_{2c} + \partial_{t_1} y_{1c} + \partial_{t_2} y_{0c} = -i\alpha y_{2t} + i\alpha^2 y_{2c} + i\alpha^2 H \cos(\omega t_0) y_{1t} - i\alpha\beta H \cos(\omega t_0) y_{1t} + i\alpha^2 G Y_{0t} - i\beta G y_{0c}$$

The term $\partial_{t_2} Y_0$ participates in the evolution at longer times and therefore is not kept here. The equations are time-averaged in t_0 to isolate the wave interaction with the forcing, so that all linear terms in $\cos(\omega t_0)$, $\sin(\omega t_0)$ vanish. The value of Y_1 found above is substituted in the remaining expressions. This yields the slow-time variation of the contour, due to the interaction of neutral waves with the forcing

$$\frac{\partial^2 y_{0t}}{\partial t_1^2} - \alpha^2 (1 - \chi) \left[G - \frac{\alpha^2 H^2}{\omega^2}(1 - \chi/2)\right] y_{0t} = 0$$

where $\chi = \beta/\alpha$. The same equation holds for y_{0c}.

This equation is now physically interpreted:

For a supercritical steady flow ($G > 0$, $H = 0$, $\chi = 0$), adding an unsteady mean flow ($H \neq 0$ or $\chi \neq 0$) can have several effects:

- either $\chi < 1$ and the unsteady component of the flow can stabilize the vortex flow when $\alpha^2 H^2/\omega^2 > G/(1 - \chi/2)$ (the case $\chi = 0$ is the case described in [4]),
- or $\chi = 1$, and there is linear growth of y_{0t} with the slow time t_1,
- or $\chi > 1$ and stabilization will occur if $(1 - \chi/2)\alpha^2 H^2/\omega^2 < G$. Note that if $\chi > 2$, this condition is automatically satisfied.

For a subcritical steady flow ($G < 0$, $H = 0$, $\chi = 0$), adding an unsteady mean flow ($H \neq 0$ or $\chi \neq 0$) can have several effects:

- either $\chi < 1$, and the flow remains stable in the presence of an oscillatory component of the baroclinic velocity (this includes the equal layer thickness case $\xi = 0$ for which $\chi = 0$; a specific study of this case is given below),
- or $\chi = 1$, and there is linear growth of y_{0t} with the slow time t_1,
- or $1 < \chi \leq 2$ and then the flow becomes unconditionally unstable on long times,
- or finally $2 < \chi$, and the flow will be unstable on long times if $2|G| > (\chi - 2)\alpha^2 H^2/\omega^2$.

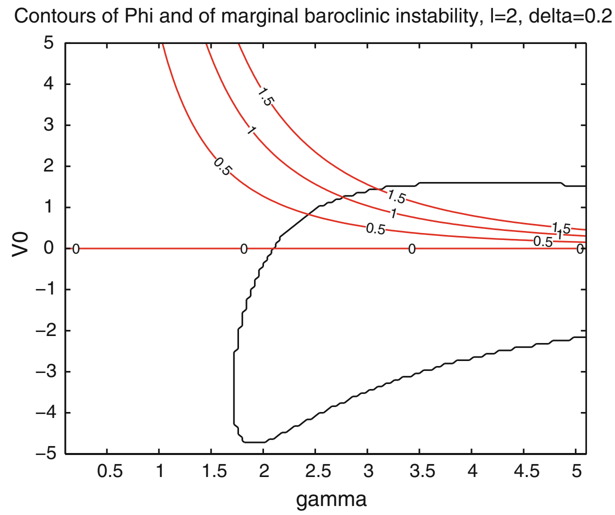

Fig. 1 Contours of ϕ (with values) in the (γ, V_0) plane, superimposed on the marginal curve for baroclinic instability (*solid black line* without value)

To relate χ to the vortex parameters, we write $\chi = \beta/\alpha = 2d_1/(d_0 - a_0) = 2/(1+\phi)$, where

$$\phi = \frac{V_t(1)\left(\frac{1}{l} - 2I_l(\gamma)K_l(\gamma)\right)}{V_c^0\xi\left(1 - \frac{I_l(\gamma)K_l(\gamma)}{I_1(\gamma)K_1(\gamma)}\right)}$$

This formulation excludes the equal layer case $\xi = 0$. We can now state that the condition $\chi = 1$ is $\phi = 1$, and the condition $\chi = 2$ is $\phi = 0$. For $l = 2$, $\delta = 0.2$ ($\xi \sim 1.8$), the values of ϕ are plotted in the (γ, V_0) plane (where $V_0 = V_t(1)/V_c^0$, see Fig. 1). The marginal curve of baroclinic instability is superimposed. Since the isolines $\phi = 0$ and $\phi = 1$ cross the marginal curve, stabilization of supercritical flows and destabilization of subcritical flows can occur.

In the case of equal layer thicknesses, $\xi = 0$ (and thus $\chi = 0$); then, destabilization of a subcritical flow cannot occur with the forcing used above. But Pedlosky and Thomson [4] showed that a weak, low frequency forcing of the type $h(t) = \varepsilon^2(G + H\cos(\omega t_1))$ can lead to such a destabilization. Note that $\omega t_1 = \varepsilon\omega t_0$, hence the low frequency forcing. We adapt their calculation to the present case. At zeroth order in ε, the equations are unchanged from the previous case. At first order in ε, they are

$$\begin{aligned}
\partial_{t_0} y_{1t} + \partial_{t_1} y_{0t} &= i\alpha y_{1t} - i y_{1c} \\
\partial_{t_0} y_{1c} + \partial_{t_1} y_{0c} &= -i\alpha y_{1t} + i\alpha^2 y_{1c}.
\end{aligned}$$

The same calculation as for zeroth order shows that $\partial_{t_0} Y_1 = 0$.

Finally, at second order, the equations are

$$\begin{aligned}
\partial_{t_0} y_{2t} + \partial_{t_1} y_{1t} + \partial_{t_2} y_{0t} &= i\alpha y_{2t} - i y_{2c} - iH\cos(\omega t_1) y_{0c} - iG y_{0c} \\
\partial_{t_0} y_{2c} + \partial_{t_1} y_{1c} + \partial_{t_2} y_{0c} &= -i\alpha y_{2t} + i\alpha^2 y_{2t} + i\alpha^2 H\cos(\omega t_1) y_{0t} + i\alpha^2 G y_{0t}
\end{aligned}$$

Again, we exclude the term $\partial_{t_2} Y_0$ which participates in the evolution at longer times, we time-average the equations in t_0, we differentiate the first order equations in t_1 and we substitute $\partial_{t_1} Y_1$ from the second order equations. This leads to

$$\frac{\partial^2 y_{0t}}{\partial t_1^2} - 2\alpha^2\left[G + H\cos(\omega t_1)\right] y_{0t} = 0$$

This equation is a Mathieu equation which is integrated numerically. The resonant pulsation $\omega = 2\sqrt{-2\alpha^2 G}$ is chosen. The results are shown on Fig. 2 for $H = 0$ (non amplified oscillation) and for $H = 0.1$ (parametrically amplified oscillation). Thus, contour perturbations on a baroclinically stable vortex can be amplified via resonance with a low frequency forcing.

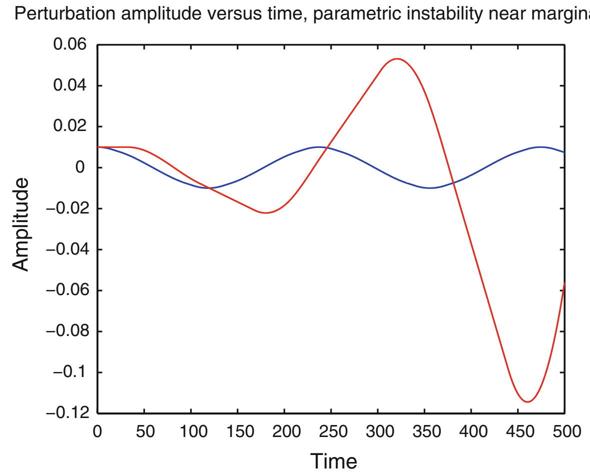

Fig. 2 Parametric amplification of the contour perturbation for $\gamma = -2.0$, $G = -0.1$, $H = 0.1$, $y_{0t}(t_1 = 0) = 0.01$, $\mathrm{d}y_{0t}t_1(t_1 = 0) = 0.0$ and at marginality of baroclinic instability. The sinusoidal variation corresponding to the steady case $H = 0.0$ is shown as reference

4 Conclusions

For a baroclinic vortex in a two-layer quasi-geostrophic flow, the interaction of neutral Rossby waves, associated with the vorticity jump at the vortex periphery, with the oscillatory component of the mean baroclinic velocity, can lead to parametric resonance, near marginality of baroclinic instability. Under given conditions on the steady and oscillatory mean velocities, parametric effects can stabilize vortex flows, which would otherwise be baroclinically unstable. Conversely, parametric resonance can destabilize subcritical baroclinic flows, if layer thicknesses are different. If layer thicknesses are equal, it was shown that a low frequency oscillation of the baroclinic mean flow can destabilize a subcritical vortex flow.

For application to the ocean, the present study should be extended to more complex flows.

Acknowledgments XC acknowledges fruitful discussions with Dr Francis Poulin during the course of this work. The authors thank a referee for a thorough analysis of the paper and for fruitful suggestions.

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