MARINE PHYSICS

Stationary Motion of a Stratified Fluid above a Rough Bottom (Geostrophic Approximation on the β-Plane)

V. F. KOZLOV AND M. A. SOKOLOVSKIY

The problem of the motion of stably stratified zonal fluid flow above an isolated disturbance of bottom topography is formulated in the geostrophic approximation on the β-plane. The solution is constructed in the form of a series in powers of the logarithm of local depth. It is shown that in the case of a linear stratification a three-dimensional cyclonic (anticyclonic) eddy forms above the obstacle. The intensity of that eddy decreases (increases) with depth for an eastward (a westward) flow advancing at a constant rate.

1. In the past decade interest in the problem of the motion of a rotating stratified fluid above an uneven bottom has grown greatly from the standpoint of geophysical hydrodynamics. An overwhelming majority of studies dealing with this problem is concerned with quasigeostrophic inertial models in the presence of a constant Coriolis parameter (a review of the problems solved from this premise is contained in [2]). Far fewer studies have dealt with the β-plane, i.e. with allowance for the latitudinal variation in Ω. McCartney [2] examined a two-layer inertial quasigeostrophic model for the case of a bottom obstacle represented by a straight circular cylinder. In a later study the same author [3] investigated the case of a continuous but weak stratification, in which the density equation does not allow for vertical advection. When the problem is so stated, the field of currents proves to be independent of the vertical stratification of the incident flow. Janowitz [4] examines an analogous model, and in his density equation he linearizes the term with vertical advection relative to the vertical gradient of the unperturbed density field and applies the bottom boundary conditions to an unperturbed horizontal plane. By way of a specific example, he examines the flow of a kinematically homogeneous and linearly stratified current around a meridional underwater range. And finally there is the recent paper by Johnson [5] who investigates the case of extremely strong stratification within the framework of the quasigeostrophic inertial model.

All the studies described above considered forms of relief admitting discontinuities in bottom slopes. Given such conditions, to eliminate the attendant singularities in the velocity field, the model had necessarily to be quasigeostrophic (inertial or viscous).

This paper examines a purely geostrophic model on the β-plane. This signifies that the bottom $z = H(x, y)$ represents a sufficiently smooth surface and that all the types of Rossby waves are filtered out. The geometry of the problem is shown schematically in Fig. 1. The starting assumptions are the customary ones in the theory of the ideal thermocline: geostrophic balance, hydrostatic balance, incompressibility, and stationary conditions. The rigid-lid condition is specified for the upper boundary. The notation and the directions of coordinate axes are conventional.

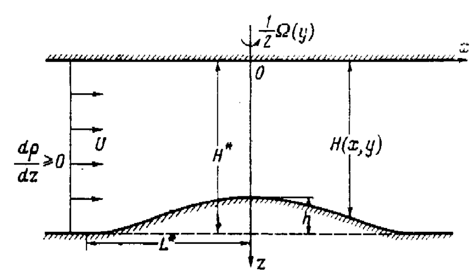

Fig. 1. Geometry of the problem.

2. We will consider a class of motions for which the hydrodynamic pressure field admits the representation $p=\rho_0 gz+\delta^* gH^*\cdot Q(\xi,\eta,\zeta)$ with the dimensionless variables $\xi=\ln[H^{*-1}H(x,y)]$; $\eta=\ln[\Omega^{*-1}\Omega(y)]$, $\zeta=z[H(x,y)]^{-1}$, where the asterisks denote the characteristic values of the depth, the Coriolis parameter, and the density perturbation; ρ is a constant. It is convenient to introduce the ancillary function

$$S(\xi,\eta,\zeta)=\int_0^{\zeta}(Q_{\xi}-\zeta Q_{\zeta})\,d\zeta. \qquad (1)$$

From equations of geostrophic motion, continuity, and hydrostatics it is readily possible to derive the formulas for the three velocity vector components and for

0001-4370/79/1804-0001$09.00/1

density [1]:

$$u = -(\rho_0\Omega)^{-1}\delta^* gH^*(H^{-1}H_y S_\zeta + \Omega^{-1}\Omega_y Q_\eta), \quad (2)$$

$$v = (\rho_0\Omega)^{-1}\delta^* gH^* H^{-1}H_x S_\zeta, \quad (3)$$

$$w = (\rho_0\Omega^2)^{-1}\delta^* gH^* H_x\Omega_y S, \quad (4)$$

$$\rho = \rho_0 + H^{-1}\delta^* H^* Q_\zeta. \quad (5)$$

The substitution of these formulas into the incompressibility condition results in the equation

$$L(Q,\ S) \equiv Q_\eta S_{\zeta\zeta} - Q_{\eta\zeta}S_\zeta - Q_{\zeta\zeta}S = 0, \quad (6)$$

which should be integrated for the boundary conditions

$$S = 0,\quad \zeta = 0;\quad S = -Q_\eta,\quad \zeta = 1. \quad (7)$$

It is also necessary to specify an additional boundary condition in the region with unperturbed bottom relief

$$Q = Q_0(\eta,\ \zeta),\quad \xi = 0. \quad (8)$$

Thus, in addition to kinematic boundary conditions (7), which represent the rigid-lid condition and the condition of stall-free flow around the bottom, respectively, the structure of the fields of density and zonal motion at H = H* should also be specified.

We will seek the solution of problem (1), (6)-(8) in the form of the power series

$$Q = \sum_{n=0}^{\infty} Q_n(\eta,\ \zeta)\xi^n,\quad S = \sum_{n=0}^{\infty} S_n(\eta,\ \zeta)\xi^n. \quad (9)$$

Substituting these formulas into the equations and boundary conditions, we derive the sequence of problems linear with respect to S_n:

$$L(Q_0,\ S_n) = G_n,\quad n = 0,\ 1, \ldots \quad (10)$$

$$S_n = 0,\quad \zeta = 0;\quad S_n = -Q_{n,\eta},\quad \zeta = 1, \quad (11)$$

where $G_n = \sum_{m=0}^{n-1} L(Q_{n-m},\ S_m)$, $n \geqslant 1$ and $G_0 = 0$; in this connection, by virtue of (1), the formula

$$Q_{n+1} = \frac{1}{n+1}(S_{n,\zeta} + \zeta Q_{n,\zeta}),\quad n = 0,\ 1, \ldots \quad (12)$$

holds true.

The question of an unambiguous solvability of problems (10), (11), as well as of the convergence of series (9), is closely related to analysis of the corresponding Sturm-Liouville problem, whose investigation presents considerable difficulty in the presence of arbitrary $Q_0(\eta,\ \zeta)$. Henceforth we will assume an unambiguous solvability of problems (10), (11) and, in particular, we will assume that condition $Q_{0,\eta} \neq 0$ is satisfied.

Suppose $M(\eta,\ \zeta)$ and $N(\eta,\ \zeta)$ are particular solutions of homogeneous equation (10) satisfying the boundary conditions $M(\eta,\ 0) = 0$, $M(\eta,\ 1) = 1$ and $N(\eta,\ 0) = 1$, $N(\eta,\ 1) = 0$. We derive Green's function

$$K(\eta,\ \zeta;\ \zeta') = \frac{1}{W(\eta,\ \zeta')} \begin{cases} M(\eta,\ \zeta)N(\eta,\ \zeta'), & \zeta \leqslant \zeta', \\ M(\eta,\ \zeta')N(\eta,\ \zeta), & \zeta > \zeta', \end{cases}$$

where $W(\eta,\ \zeta) = MN_\zeta - M_\zeta N$ is the Wronskian. Then the solution of problem (10), (11) is written as

$$S_n(\eta,\ \zeta) = -Q_{n,\eta}(\eta,\ 1)\,M(\eta,\ \zeta) + \int_0^1 K(\eta,\ \zeta;\ \zeta')\frac{G_n}{G_{0,\eta}}(\eta, \zeta')\,d\zeta'. \quad (13)$$

Formulas (9), (12), (13) exhaust the formal algorithm. On constructing Q and S in accordance with formulas (2)-(5) we determine the velocity and density fields in the presence of an arbitrary sufficiently smooth perturbation of the bottom relief. The computation of the integral characteristics of the flow (integral streamfunction, potential energy, mass content) presents no difficulties; e.g. for integral streamfunction we have

$$\Psi(x,\ y) = \frac{\delta^* gH^{*2}}{\rho_0\Omega^*}\sum_{n=0}^{\infty}\Phi_n(\eta)\xi^n, \quad (14)$$

where

$$\Phi_0(\eta) = \int_0^\eta e^{-\eta}\left(\int_0^1 Q_{0,\eta}(\eta,\ \zeta)\,d\zeta\right)d\eta + \text{const},$$

$$\Phi_{n+1}(\eta) = -\frac{e^{-\eta}}{n+1}\sum_{m=0}^{n}\frac{1}{m!}Q_{n-m,\eta}(\eta,\ 1),\quad n \geqslant 0.$$

3. We will describe an example of implementation of the algorithm given above for the case in which the stably stratified unperturbed flow represents a zonal current with a constant velocity U. Under these conditions $Q_0(\eta,\ \zeta) = -0{,}5\,k^{-2} + R(\zeta)$, $k = k_0 e^{-\eta}$, where, in accordance with (5), the function $R(\zeta)$ characterizes the stratification of the unperturbed flow. The principal external parameter of the problem is

$$k_0 = \Omega^{*-1}[\delta^* gH^*\beta(\rho_0 U)^{-1}]^{-1/2}.$$

Operator (6) becomes $L(Q_0, S) = -k^2 S_{\zeta\zeta} - R''(\zeta)S$. Consider a case in which the fundamental system of equations can be expressed through elementary functions. Suppose the vertical density distribution in unperturbed flow ($\xi = 0$) obeys the hyperbolic law

$$\rho = \rho_0 + \delta^*\frac{\zeta - 1}{1 + \gamma\zeta},\quad \frac{d\rho}{dz} = \frac{\delta^*}{H^*}\frac{1 + \gamma}{(1 + \gamma\zeta)^2},\quad \zeta = \frac{z}{H^*}.$$

The parameter $\gamma > 0$ determines the degree of the profile's deviation from the linear, which is present up to the limit of $\gamma \to 0$. It is readily verified that for this case the function $R(\xi)$ can be taken as

$$R(\zeta) = -\gamma^{-2}(1 + \gamma)\ln(1 + \gamma\zeta) + \gamma^{-1}\zeta + 0{,}5. \quad (15)$$

The fundamental system of solutions and the Wronskian are written as

$$M(\eta,\ \zeta) = \left(\frac{1 + \gamma\zeta}{1 + \gamma}\right)^{1/2}\frac{\sin[\omega\ln(1 + \gamma\zeta)]}{\sin[\omega\ln(1 + \gamma)]}, \quad (16)$$

$$N(\eta, \zeta) = (1+\gamma\zeta)^{1/2} \frac{\sin\left[\omega \ln\left(\frac{1+\gamma}{1+\gamma\zeta}\right)\right]}{\sin[\omega \ln(1+\gamma)]}, \tag{17}$$

$$W(\eta, \zeta) = -\frac{\gamma\omega}{(1+\gamma)^{1/2} \sin[\omega \ln(1+\gamma)]},$$

$$\omega = \frac{1}{\gamma}\left|k^2(1+\gamma) - \frac{\gamma^2}{4}\right|^{1/2}. \tag{18}$$

When $k^2 < 1/4\,\gamma^2(1+\gamma)^{-1}$ the circular sines in formulas (16)-(18) should be substituted with their hyperbolic counterparts. Elementary analysis demonstrates that (16)-(18) represent continuous functions of the parameter k^2, expecting the points $\omega \ln(1+\gamma) = n\pi,\ n \geqslant 1$ at which solution does not exist. There is reason to believe that the requirement $Q_n \neq 0$ will be satisfied only by the solutions derived for $-\infty < k^2 < \gamma^2(1+\gamma)^{-1}\{1/4+\pi^2[\ln\cdot(1+\gamma)]^{-2}\}^{1/2}$. When $\gamma \to 0$ this condition turns into B $-\infty < k^2 < \pi^2$, which in the quasi-geostrophic approximation precludes internal Rossby waves [4].

For linear stratification ($\gamma \to 0$) relations (15)-(18) are rewritten as

$$R(\zeta) = \frac{1}{2}(\zeta - 1)^2, \quad M = \frac{\sin k\zeta}{\sin k},$$

$$N = \frac{\sin k(1-\zeta)}{\sin k}, \quad W = -\frac{k}{\sin k}.$$

respectively. For this case the first three terms of expansions (9) were computed, and the findings were applied to the bottom relief

$$H(x, y) = \begin{cases} 1 - h\left(1 - \frac{x^2+y^2}{L^{*2}}\right)^2, & x^2+y^2 \leqslant L^{*2} \\ 1, & x^2+y^2 > L^{*2}. \end{cases}$$

When h > 0 we have an axisymmetric underwater elevation with a maximum height of hH*, whereas when h < 0 we are dealing with an underwater trough. Sample calculations were performed for the following values of parameters: $\varphi^* = 30°$, $L^* = 100$ km, $H^* = 4$ km, $|U| = 10$ cm·sec^{-1}, h = 0.025, $k_0 = \pi/2$, which corresponds to the density perturbation $\delta^* = 1{,}65\cdot 10^{-3}$ · cm^{-3} and the Rossby number $\varepsilon = |U|/L^*\Omega^* = 1{,}37\cdot 10^{-2}$.

Figure 2 presents the computed streamlines (isobars) of horizontal motion at depth levels z/H* = 0 (a), 0.6 (b), 1-h (c) for an unperturbed eastward current (U > 0); in this connection, by virtue of symmetry, only the right half of the pressure field is shown. It is clearly seen that a perturbation of the relief manifests itself most strongly at the surface and very weakly at the level of the crest of the underwater elevation. Above the northern flank there forms a cyclonic eddy formation which weakens with depth and vanishes at some intermediate depth level (z/H* > 0.6).

Figure 3 shows the patterns of isobars at the same depth levels for the case of a westward current (U < 0). Here the effect of relief and stratification is of a qualitatively different nature. The anticyclonic eddy above the northern flank reaches its maximum development at the bottom and, gradually weakening

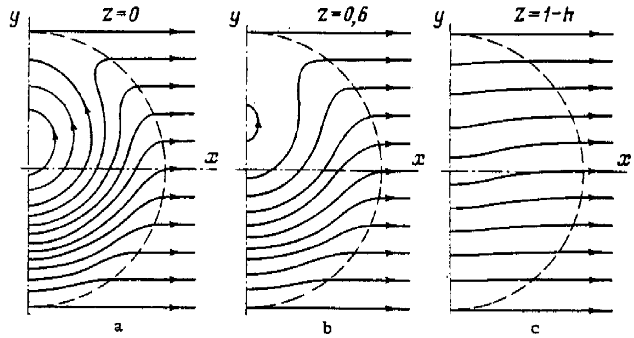

Fig. 2. Easterly current (U > 0) above an axisymmetric underwater elevation having the height h. Streamlines at dimensionless depth levels.

a) z = 0; b) z = 0.6; c) z = 1-h.

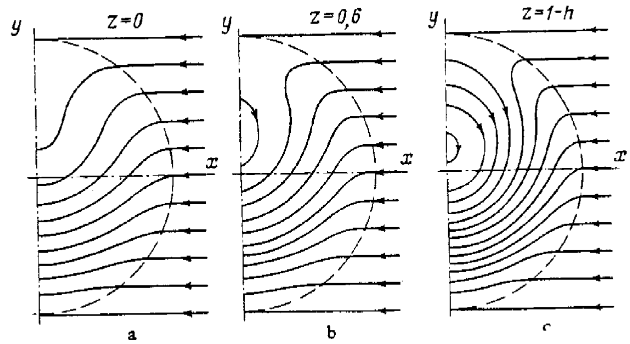

Fig. 3. As in Fig. 2, but westerly flow (U < 0).

toward the surface, disappears when z/h* < 0.4. In both cases the pattern of the integral circulation, derivable from formula (14) is similar to the corresponding patterns of horizontal flow at the intermediate depth level z/H* = 0.6. The characteristic variations in velocity then are of the order of U.

The differences in the response of the geostrophic stratified flow depending on its direction may be explained as follows. From the structure of formulas (2) and (3) it ensues that horizontal motion consists of two components: the planetary zonal motion proportional to $\Omega_y \equiv \beta$ and the topographic motion along the isobaths proportional to the degree of bottom slope, with the latter — when $S_\zeta > 0$ — being anticyclonic above the elevations and cyclonic above the troughs. When the polarity of S_ζ is reversed, the topographic motion reverses its direction. As shown by calculations [1], when $\xi < 0$ (H < H*) and U > 0, S_ζ is positive and decreases with depth, but when U < 0 it becomes negative and increases in absolute value with depth. This signifies that the intensity of topographic motion in an eastward incident flow weakens with depth, whereas in a westward flow it gains in strength. And indeed we observe manifestations of this effect in Figs. 2 and 3. Note that in the presence of Rossby waves the stratification in quasigeostrophic inertial models weakens the perturbations with increasing distance from the bottom regardless of the direction of flow [2, 4].

Thus, the interaction between a geostrophic stably stratified flow and a local depth perturbation may result in three-dimensional eddy formations whose intensity weakens with depth for eastward incident flow and increases in depth for westward incident flow.

Of course, the example examined in this paper represents for the limiting case of $\gamma \to 0$ an extremely rough approximation from the standpoint of applicability to the ocean. When the function $R(\zeta)$ is of a more general form (15), the proposed model may be used to investigate the effect of a mesoscale topographic perturbation on near-zonal currents (e.g. the antarctic circumpolar current).

In conclusion, the authors are indebted to A.A. Skotnikov and N.A. Kashirikhin for performing the necessary calculations on a computer.

REFERENCES

1. Kozlov, V.F. Geostrophic motion of a stratified fluid above a rough bottom. Izv. Akad. nauk SSSR. Fizika atmosfery i okeana, 13, No. 9. 1977.
2. McCartney, M.S. Inertial Taylor columns on a beta-plane. J. Fluid Mech., 68, 1975, p. 1.
3. McCartney, M.S. The interaction of zonal currents with topography with applications to the Southern Ocean. Deep-Sea Res., 23, No. 5, 1976.
4. Janowitz, G.S. The effect of bottom topography on a stratified flow in the beta-plane. J. Geophys. Res., 80, No. 30, 1975.
5. Johnson, E.R. Stratified Taylor columns on a beta-plane. Geophys. Astrophys. Fluid Dyn., 9, No. 2, 1977.

Far Eastern State University

Pacific Oceanological Institute
Far Eastern Scientific Center
USSR Academy of Sciences

Received November 1, 1977