ORIGINAL ARTICLE

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On instability of elliptical hetons

Abstract Using the method of contour surgery, we examine the evolution of an initially vertically aligned elliptical heton. A classification of quasi-stable and unstable regimes for the case of two-layered vortex structure is suggested.

Keywords Elliptical vortex · Vortex instability · Heton

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1 Introduction

The classic Kirchhoff [1] solution for the elliptical vortex patch with semi-axes a and b and vorticity ω, which rotates as a solid body with a constant angular velocity $\Omega = \omega ab/(a + b)^2$ is known from 1876. Seventeen years later Love [2] showed that for $\chi = a/b > 3$ this stationary solution is unstable. Recently, Mitchell and Rossi [3] have given a complete quantification of the regimes of linear and non-linear instability of the elliptical vortex with respect to the parameter χ.

  Kirchhoff solution had multiple generalizations. In particular,

– Chaplygin [4,5], and later Kida [6], Dritschel [7], Dhanak and Marshall [8], Legras and Dritschel [9] and others have shown that the introduction of an ambient velocity field, linearly dependent on the coordinates (this is analogous to the affine transformation of coordinates, converting ellipses into ellipses), allows to

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construct a solution, defining the behaviour of a pulsating vortex, which rotates with variable velocity, and it is a time-dependent value of χ.

– Polvani and Flierl [10] have introduced a notion of a generalized Kirchhoff vortex for a system of N embedded one into other elliptical patches. They have studied the stability of such a solution, and constructed diagrams of stable and unstable states in the space of external geometric parameters.
– Kozlov [11] has summarized the problem of an elliptical vortex when the effect of the "entrainment" due to the introduction of an "effective" bottom friction is taken into account. This mechanism initiates cyclonic-anticyclonic asymmetry (when ω changes its sign), which is observed in ocean and atmosphere conditions. In [11] the author provides the conditions under which the model gives the particular cases of Kirchhoff, Chaplygin and Kida.

The above works investigate the dynamics of elliptical vortex patches in a homogeneous non-rotating fluid. These results may be applied also to the case of rotating fluid under the condition of geostrophic balance (when the hydrodynamic pressure plays the role of the stream function).

In this paper, in the framework of quasi-geostrophic approximation [12] we examine the problem on the behaviour of two vortex patches in a stratified rotating fluid, which consists of two non-mixing layers of equal thickness. It is supposed also that in the initial moment elliptical vortices of the upper and bottom layers are kinematically identical, and are located strictly one over the other, and they have opposite signs of ω. Such a two-layer vortex represents a *heton* [13].

In [14], a stationary solution in the form of an axially symmetrical heton has been investigated to be stable with respect to small harmonic disturbances of the boundary at $\chi = 1$. There is shown that at $\gamma = R/R_d > 1.705$ (R is a characteristic horizontal scale, R_d is the internal deformation Rossby radius [12]) may appear unstable modes. Thus, it is established that even the circular heton may be unstable with respect to the degree of the stratification. Let us note that similar results have been obtained independently in [15] and [16].

2 Numerical modelling of evolution

For elliptical vortex patches ($\chi > 1$) it is possible to carry out only numerical study of stability. It should be noted, that the combination of the vertically aligned non-circular vortex patches of opposite cyclonic vorticity in two-layer fluid, strictly speaking, does not satisfy a stationary state. In this case the quasi-elliptical pulsating vortex patches rotate in different directions, and each of them induces an external field for its partner, which changes periodically both in time and in space. Therefore, we will interpret the initial state represented as the combination of two vertically aligned ellipses as some instantaneous disturbed quasi-stationary state. Below, we give the results of numerical experiments for investigating the evolution of such vortices. We used the two-layer version of the Contour Dynamics Method (surgery) [14,17]. Before the discussion of the results we would like to note that $R_d = \sqrt{g'h}/2f$ ($g' = g\Delta\rho/\rho_0$, g is the acceleration of gravity, $\Delta\rho$—difference of fluid densities in the bottom and upper layers, h is characteristic vertical scale, ρ_0 the mean value of the density, f is Coriolis parameter equal to a doubled rotation velocity of the whole plane). Thereby, the parameter γ may be treated in two ways: as a stratification parameter ($\gamma \ll 1$ and $\gamma \gg 1$ being the limits of the strong and weak stratification, correspondingly), and as a geometrical characteristic of the typical size of the vortex with respect to the radius of deformation.

2.1 General results

In Fig. 1, a diagram of states of the initially elliptical heton in the space of (γ, χ) variables represents the result of numerous calculations. One can see areas of existence both of stable quasi-periodic solutions, and of the solutions with different types of instability. Summary results are the following:

– In case of a *strong stratification* (or *small relative sizes* of vortex patches) the *barotropic* type of interaction is predominant, when the vortices from the different layers weakly interact one with the other, and their evolution occurs essentially by the self-interaction.
– When the eccentricity of the initially elliptical vortices grows, in the frames of this class of motions, there takes place a consecutive transition from the quasi-stable states of the system to the unstable ones with non-symmetrical division of the vortex patches; and further to symmetrical division into two parts. In the same time the area of the quasi-barotropic motions shrinks.
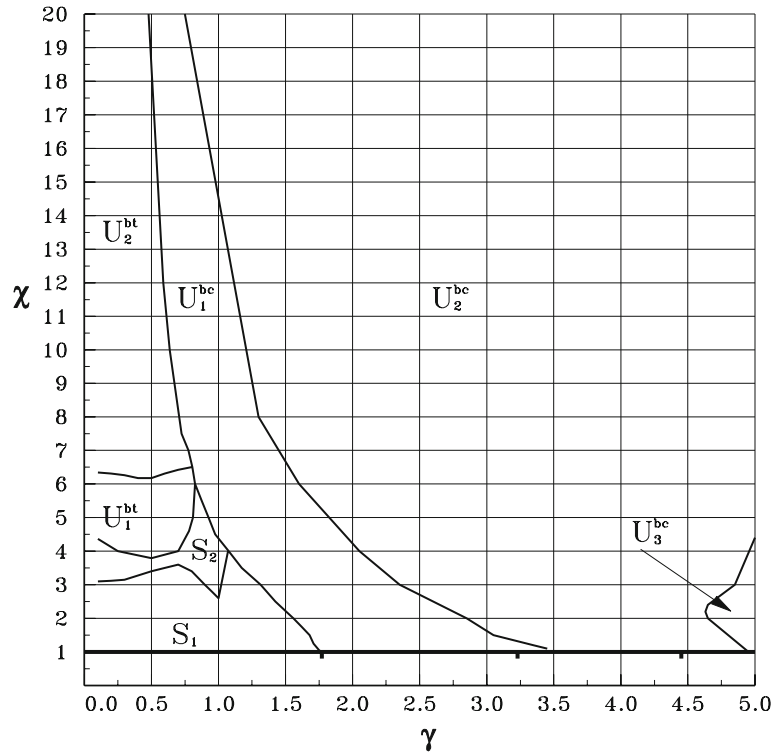
Fig. 1 Diagram of possible states of an elliptical heton with the vertical axis in a rectangular area $\gamma \in (0, 5]$, $\chi \in [1, 20]$. Symbols for sub-areas: S_1, stable states; S_2, quasi-stable states after a partial loss of the mass because of dropping of vortex threads; U_1^{bt}, decay of the elliptical patches into non-symmetric parts because of the barotropic instability; U_2^{bt}, decay of elliptical patches into symmetric parts because of the barotropic instability; U_1^{bc}, decay of the elliptical patches into two hetons with tilted axes running away in the opposite directions because of the barotropic instability; U_2^{bc}, cascade baroclinic instability with running away of a series of hetons along the main ellipse axis; U_3^{bc}, cascade baroclinic instability with running away of hetons along both ellipse axes. Notches on the *solid straight line* $\chi = 1$ divide the following areas for axially symmetrical heton: stability, and unstable mode with $m = 2$, $m = 3$ and $m = 4$ (from left to right)

– In case of a *moderate stratification*, a *baroclinic* type of interaction predominates with the realization of the vortex structure decay into two running in the opposite directions two-layer pairs with tilted axes. With the growth of ellipticity, the domain of existence with respect to the parameter γ of this class of motions also significantly shrinks.
– In case of a *strong stratification*, the main mechanism of the development of the *cascade instability*, resulting in the further running away of more than two two-layer pairs.

2.2 Analysis of realizations of different regimes

During our computation, the potential vorticity of the upper-layer vortex patch was taken to be positive, and of the lower-layer one is negative. The absolute value of ω was defined in each of the experiments by in such a way that a half of the rotational period of a fluid particle on the contour of a circular heton at the given value of γ would correspond to a dimensionless unit of time.

Figure 2 shows the examples of calculations of the elliptical hetons' evolution in the vicinity of the boundary of areas S_1 and U_1^{bc} at the fixed values of the parameter $\chi = 2$. In the first case a quasi-stable rotation of pulsating vortex patches in the upper and lower layers (barotropic type of interaction) occurs in the opposite directions. Vortex patches take a shape close to the elliptical one in the moment of the mutual overlapping (when the angular velocity of rotation is maximum) and the dumbbell shape when the conventional semi-axes have normal position (angular velocity is minimum in those moments). Note, that these effects—irregularities of rotation and deformation of contours—are expressed the stronger, the closer is the correspondent point of the plane (γ, χ) to the boundary of the stability and instability areas. In the second case, due to the mechanism
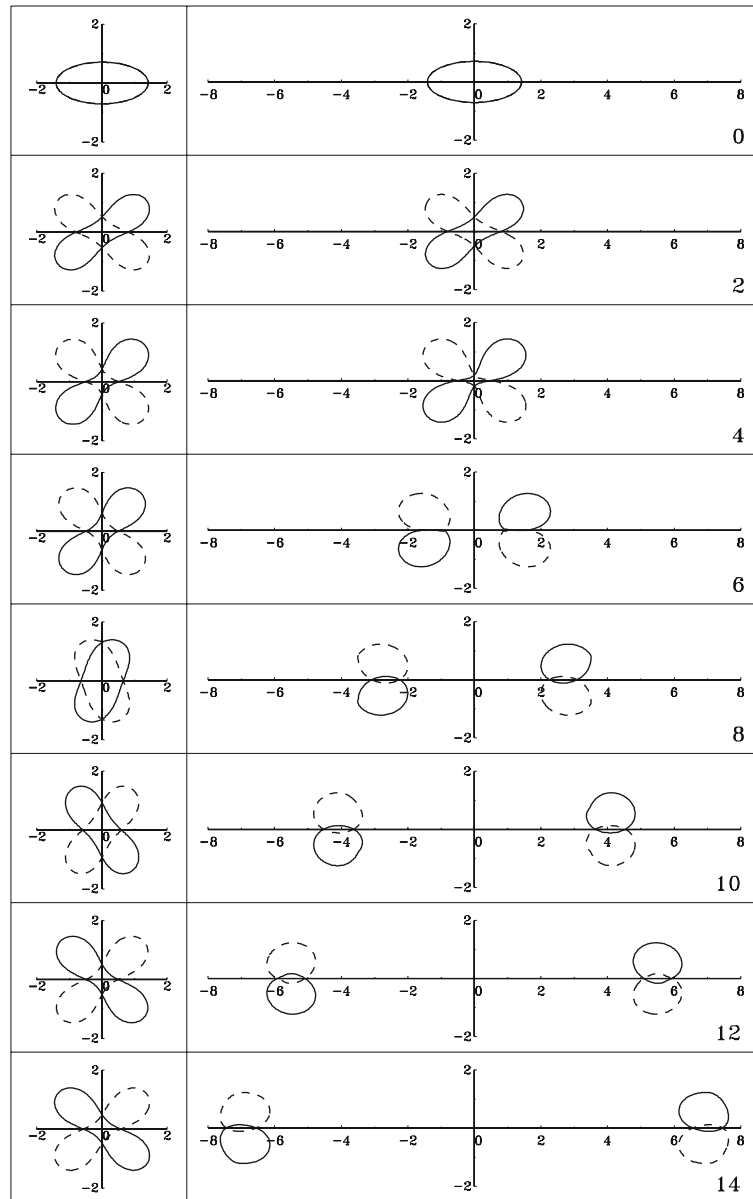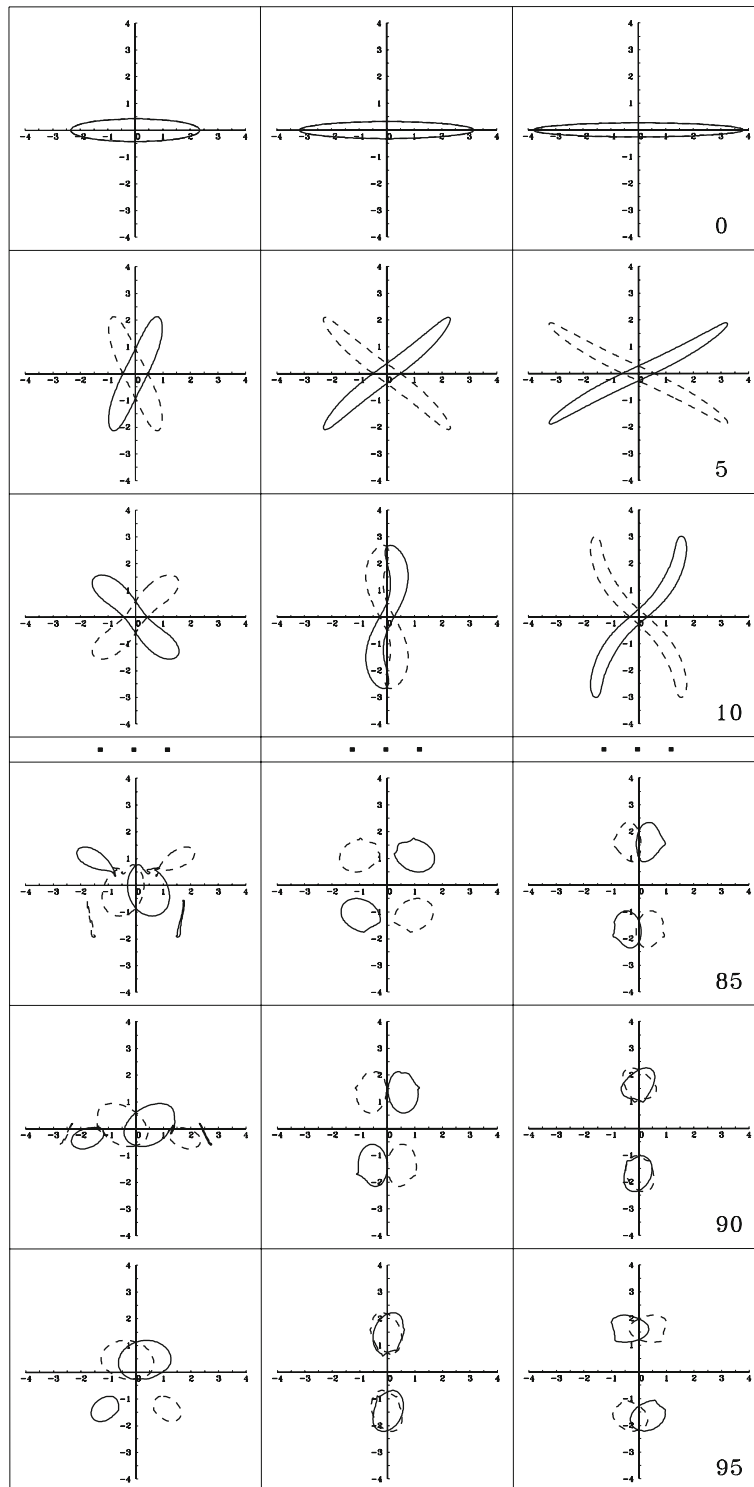
Fig. 2 Synchronous configurations of the quasi-stable (*left* at $\gamma = 1.56$) and unstable (*right* at $\gamma = 1.57$) elliptical hetons at $\chi = 2$ at the indicated moments of dimensionless time. *Solid* and *dashed* lines depict contours of the vortex patches in the upper and bottom layer, correspondingly

of baroclinic instability, the initial structure decays into two running away two-layer pairs with tilted axes. Let us note that there exists qualitative analogy of these variants with the case of two aligned point hetons [13], when the distance between them is either small, or large, correspondingly.

Figure 3 demonstrates the evolution on the non-linear stage of the barotropic instability for the non-symmetrical (U_1^{bt}, left column) and symmetrical (U_2^{bt}, central and right columns) types at a fixed value of γ and different values of χ at the initial stage (before the separation of the vortex patches) and at the quasi-stationary stage of rotating movements of the vortices, that have being formed as a result of the decay of the initial vortex structure. The fact of the non-symmetrical decay of elliptical barotropic vortices at relatively small values of χ has been established for the first time in the numerical calculations by Kozlov and Makarov [18].

In Fig. 4 we illustrate the possibility of transition between the regimes of the cascade instability U_2^{bc} and U_3^{bc}, when the parameter γ increases. Note, that the initial evolution in the both variants runs identically, but

Fig. 3 Synchronous configurations of the non-symmetrical (*left* at $\chi = 5.5$) and symmetrical (*central* at $\chi = 10$ and *right* at $\chi = 14.5$) unstable elliptical hetons when $\gamma = 0.25$ at the indicated dimensionless times

at the instant $t = 2$, the central part in the second case begins by now to stretch in the vertical direction, what ensure the further running away of small vortex patches in the opposite directions of the y axis. This is the manifestation of the fourth mode formation.
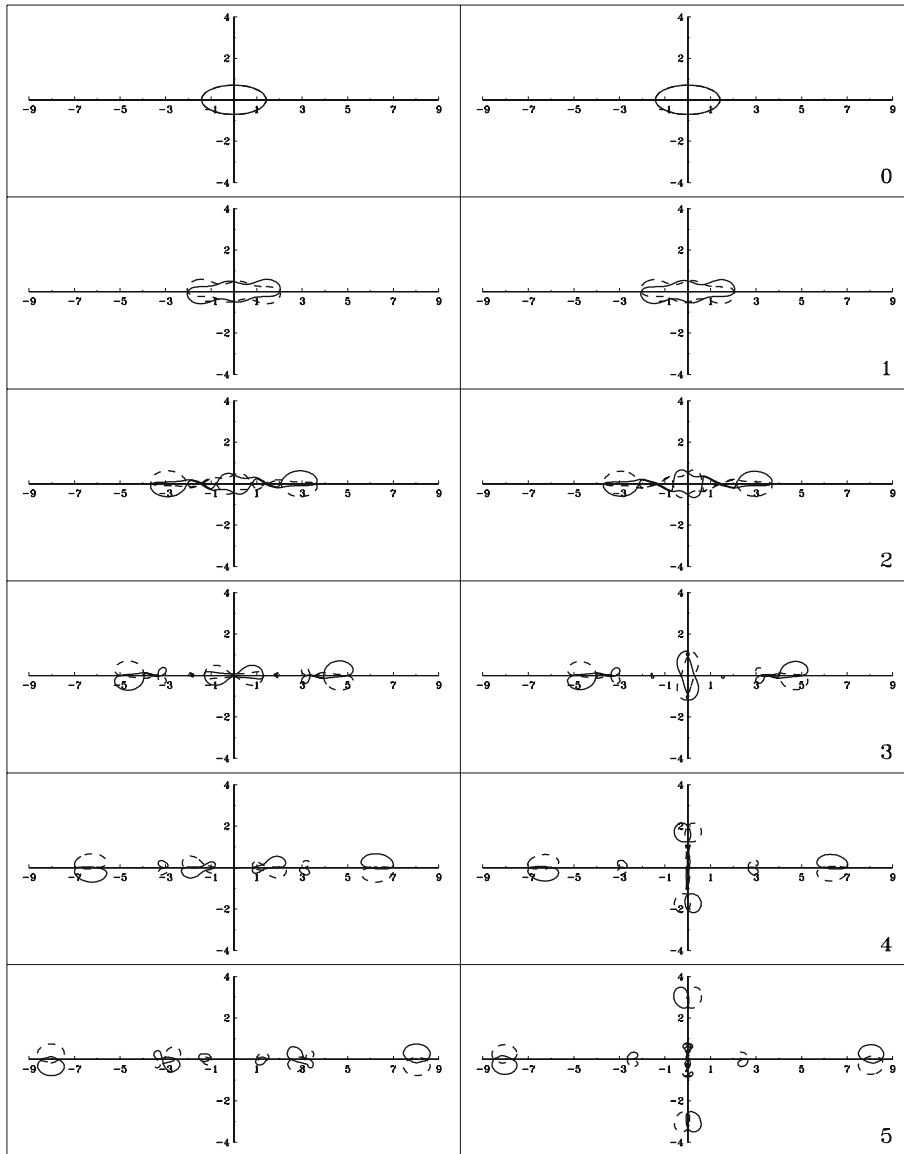
Fig. 4 Synchronous configurations of unstable elliptical hetons with vertical axes at $\chi = 2$ and $\gamma = 4.55$ on the *left* (type U_2^{bc}) and $\gamma = 4.57$ on the *right* (type U_3^{bc})

Note that Figs. 2 and 4 demonstrate a "robust" character of the vortex structures behaviour in dependence on the value of the parameter γ: in the vicinity of the boundary between the different regimes an insignificant change of the stratification parameter causes the transition to the qualitatively another type of the vortex patches' interaction.

3 Summary

In the present work, the classification of stable and unstable regimes of elliptical hetons is given. We have shown that the possibility exists of two types of instability—barotropic and baroclinic ones, and that a weak stratification stimulates the conditions for cascading instability.

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