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On the Stability of a System of Two Identical Point Vortices and a Cylinder

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Abstract—We consider the stability problem for a system of two identical point vortices and a circular cylinder located between them. The circulation around the cylinder is zero. There are two parameters in the problem: the added mass a of the cylinder and $q = R^2/R_0^2$, where R is the radius of the cylinder and $2R_0$ is the distance between vortices. We study the linearization matrix and the quadratic part of the Hamiltonian of the problem, find conditions of orbital stability and instability in nonlinear statement, and point out parameter domains in which linear stability holds and nonlinear analysis is required. The results for $a \to \infty$ are in agreement with the classical results for a fixed cylinder. We show that the mobility of the cylinder leads to the expansion of the stability region.

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1. INTRODUCTION

The problem of interaction of a cylinder and a vortex in an ideal fluid was posed by V. V. Kozlov. It was solved in [1, 2] and turned out to be integrable. The problem was further developed in [3, 4], where the authors presented various forms of the equations of motion for a cylinder interacting with a set of point vortices and pointed out particular integrable cases. Nevertheless, the problem of interaction of a cylinder with two point vortices is not integrable. For the general system of equations of motion of point vortices and a cylinder, there are particular Thomson's configurations where the vortices uniformly distributed over a circle move around the center of the circular cylinder.

In the classical case when the cylinder is fixed and the circulation around it is zero, the stability analysis was started by T. H. Havelock [5], who solved this problem in the linear statement. The analysis has been continued in many other works (see the survey papers [6, 11]). However, the question is not completely closed. It turned out that in the nonlinear statement the problem splits into a series of meaningful subproblems that require an individual approach. In particular, an analysis of the Taylor series of the Hamiltonian of the problem up to the forth order inclusive was needed. All resonances arising in the problem have been analyzed [17]. It turned out that two of them lead to instability, although the linear statement exhibits stability [9, 13].

The problem of the stability of the motion of Thomson's vortex configuration and a moving cylinder is new, and its analysis has not yet been performed. This problem is important because the motion of a cylinder and vortices often arises in problems of the motion of a body in a fluid and is of practical interest in connection with the development of underwater robots.

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The question of the stability of Thomson's vortex configuration and a cylinder is complicated and has no universal solution. In the present study, we restrict ourselves to the configuration of a cylinder and two vortices under the assumption of zero circulation around the cylinder. We will show that the mobility of the cylinder leads to additional stability, which is illustrated by the corresponding diagrams in Fig. 1 below.

2. EQUATIONS OF MOTION

Consider the motion of a circular cylinder that interacts with n point vortices. In [3, 4], the authors reduced the full equations of motion to a system of equations that can be represented in the complex form

$$\begin{aligned} a\dot{z}_{\mathrm{c}} &= av = -i\gamma z_{\mathrm{c}} + i\sum_{j=1}^{n}\gamma_j(\widetilde{z}_j - z_j),\\ \dot{\overline{z}}_k &= -\overline{v} + \frac{R^2 v}{z_k^2} + \frac{i\gamma}{z_k} - \frac{i\gamma_k}{z_k - \widetilde{z}_k} + i\sum_{\substack{j=1\\ j\neq k}}^{n}\gamma_j\left(\frac{1}{z_k - z_j} - \frac{1}{z_k - \widetilde{z}_j}\right), \qquad (2.1)\\ \widetilde{z}_k &= \frac{R^2}{\overline{z}_k}, \qquad k = 1,\dots,n. \end{aligned}$$

Here the complex variables $z_{\mathrm{c}} = x_{\mathrm{c}} + iy_{\mathrm{c}}$ and $z_k = x_k + iy_k$ define the position of the cylinder and point vortices, $v = v_1 + iv_2$ is the velocity of the cylinder, R is the radius of the cylinder, the constant coefficient $a > 0$ is the added mass of the cylinder, and the parameters γ and γ_j are related to the circulation of the cylinder and the intensity of the point vortices by the formulas $\gamma = \Gamma/(2\pi)$ and $\gamma_j = \Gamma_j/(2\pi)$.

System (2.1) can be rewritten as

$$\gamma\dot{z}_{\mathrm{c}} = -2iH_{\overline{z}_{\mathrm{c}}}, \qquad \gamma_k\dot{\overline{z}}_k = 2iH_{z_k}. \qquad (2.2)$$

Its Hamiltonian $H = H(\mathbf{z},\overline{\mathbf{z}})$, $\mathbf{z} = (z_{\mathrm{c}}, z_1, \dots, z_n)$, has the form

$$\begin{aligned} H = {}& \frac{1}{2}a|v|^2 + \frac{1}{2}\sum_{k=1}^{n}\bigl(\gamma_k^2\ln(|z_k|^2 - R^2) - \gamma_k\gamma\ln|z_k|^2\bigr)\\ & + \frac{1}{2}\sum_{1\le k<j\le n}\gamma_k\gamma_j\bigl(\ln|R^2 - z_j\overline{z}_k|^2 - \ln|z_j - z_k|^2\bigr). \qquad (2.3) \end{aligned}$$

System (2.2) admits the integral

$$\mathcal{I} = \gamma|z_{\mathrm{c}}|^2 - \sum_{j=1}^{n}\gamma_j|z_j|^2. \qquad (2.4)$$

Suppose that all point vortices have the same intensity γ_0:

$$\gamma_1 = \ldots = \gamma_n = \gamma_0.$$

Then, in the case of $n \ge 2$, system (2.2) has a solution

$$z_{\mathrm{c}} = 0, \qquad z_k = e^{i\omega_n t}u_k, \qquad u_k = R_0 e^{2\pi i(k-1)/n}, \qquad k = 1,\dots,n, \qquad R_0 > R, \qquad (2.5)$$

which corresponds to the stationary rotation of Thomson's vortex polygon around the cylinder at a constant angular velocity ω_n:

$$\omega_n = \frac{1}{R_0^2}\left(2\gamma + \gamma_0\left(n - 1 - \frac{2n}{1 - q^{2n}}\right)\right), \qquad q = \frac{R^2}{R_0^2}. \qquad (2.6)$$

Note that the stationary rotation regime (2.5) is certainly Lyapunov unstable for all n. Indeed, if we perturb the vortex n-gon at the initial time $t = 0$ so that it remains regular but of different size, then the further motion will still be a uniform rotation, but with a different angular velocity. As a result, however small such a perturbation is at the beginning, it becomes on the order of the polygon diameter with time. Since the Lyapunov stability is impossible in the case of permanent rotation, we refine the definitions. In this paper, by stability we mean orbital stability, and by instability, the instability of the invariant set formed by all orbits of the continuous family (2.5), which depends on the parameter $R_0 > R$.

3. STABILITY OF THE STATIONARY SOLUTION (2.5) IN THE CASE OF ZERO CIRCULATION AROUND THE CYLINDER FOR $n = 2$

Let $n = 2$. The circulation around the cylinder is zero [18], i.e., $\gamma = 2\gamma_0$. Without loss of generality, we henceforth assume that

$$\gamma_0 = 1 \qquad \text{and} \qquad R_0 = 1.$$

The change of variables

$$\begin{aligned} &z_{\mathrm{c}} = e^{i\omega_2 t} v_{\mathrm{c}}(t), \qquad && v_{\mathrm{c}}(t) = q_{\mathrm{c}} + ip_{\mathrm{c}}, \\ &z_k = e^{i\omega_2 t} v_k(t), \qquad && v_k(t) = \sqrt{1+2r_k}\, e^{i(2\pi(k-1)/n+\theta_k)}, \qquad k = 1, 2, \end{aligned} \tag{3.1}$$

where

$$\omega_2 = \frac{1-5q^2}{1-q^2}, \tag{3.2}$$

leads system (2.2), (2.3) to the perturbation equations

$$2\dot{q}_{\mathrm{c}} = \frac{\partial E}{\partial p_{\mathrm{c}}}, \qquad 2\dot{p}_{\mathrm{c}} = -\frac{\partial E}{\partial q_{\mathrm{c}}}, \qquad \dot{r}_k = \frac{\partial E}{\partial \theta_k}, \qquad \dot{\theta}_k = -\frac{\partial E}{\partial r_k} \tag{3.3}$$

with the following Hamiltonian $E = E(\boldsymbol{\rho})$, $\boldsymbol{\rho} = (q_{\mathrm{c}}, r_1, r_2, p_{\mathrm{c}}, \theta_1, \theta_2)$:

$$E(\boldsymbol{\rho}) = H(\mathbf{z}(\boldsymbol{\rho}), \overline{\mathbf{z}}(\boldsymbol{\rho})) + \omega_2\big(1 + q_{\mathrm{c}}^2 + p_{\mathrm{c}}^2 + r_1 + r_2\big). \tag{3.4}$$

The family of equilibria

$$\mathcal{C} = \big\{q_{\mathrm{c}} = p_{\mathrm{c}} = r_1 = r_2 = 0,\ \theta_1 = \theta_2\big\} \tag{3.5}$$

corresponds to the stationary solution (2.5). The stability of the continuous family of equilibria (3.5) is equivalent to the orbital stability of the stationary solution (2.5).

Let us expand the Hamiltonian $E(\boldsymbol{\rho})$ in the Taylor series in powers of $\boldsymbol{\rho}$ in the neighborhood of the zero equilibrium state:

$$E(\boldsymbol{\rho}) = E_0 + E_2(\boldsymbol{\rho}) + \dots. \tag{3.6}$$

Here the ellipsis denotes all terms of order higher than 2. The linear terms are absent: $E_1 = 0$. The quadratic form E_2 can be expressed as

$$E_2(\boldsymbol{\rho}) = \langle \mathbf{S}\boldsymbol{\rho}, \boldsymbol{\rho}\rangle, \qquad \mathbf{S} = \begin{pmatrix} \mathbf{F}_1 & 0 \\ 0 & \mathbf{F}_2 \end{pmatrix}, \tag{3.7}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product of two vectors. The matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ have the form

$$\mathbf{F}_1 = \begin{pmatrix} a_1 & b_1 & -b_1 \\ b_1 & d_1 & c_1 \\ -b_1 & c_1 & d_1 \end{pmatrix}, \qquad \mathbf{F}_2 = \begin{pmatrix} a_2 & b_2 & -b_2 \\ b_2 & -c_2 & c_2 \\ -b_2 & c_2 & -c_2 \end{pmatrix}. \tag{3.8}$$

Their coefficients are

$$a_1 = a_2 = \omega_2 + \frac{2}{a}, \qquad b_1 = \frac{1+q}{a}, \qquad b_2 = \frac{1-q}{a}, \qquad c_1 = \frac{q^2+6q+1}{4(1+q)^2} - \frac{(1+q)^2}{2a},$$
$$d_1 = \omega_2 - \frac{q^4+4q^3+22q^2+4q+1}{4(1-q^2)^2} + \frac{(q+1)^2}{2a}, \qquad c_2 = -\frac{(1-q)^2}{4(1+q)^2} - \frac{(1-q)^2}{2a}. \tag{3.9}$$

The eigenvalues of the matrix $\mathbf{S}$ are the roots of the characteristic polynomials of the matrices $\mathbf{F}_1$ and $\mathbf{F}_2$:

$$P_1(\lambda) = (\lambda - c_1 - d_1)(\lambda^2 + p_{11}\lambda + p_{10}), \tag{3.10}$$

$$P_2(\lambda) = \lambda(\lambda^2 + p_{21}\lambda + p_{20}). \tag{3.11}$$

The coefficients p_{j1} and p_{j0}, $j = 1, 2$, are defined by the formulas

$$p_{11} = c_1 - a_1 - d_1, \qquad p_{10} = -2b_1^2 + (d_1 - c_1)a_1,$$
$$p_{21} = 2c_2 - a_1, \qquad p_{20} = -2b_2^2 - 2a_1c_2. \tag{3.12}$$

The linearization matrix of the system with Hamiltonian (3.4) in a neighborhood of the zero equilibrium state has the form

$$\mathbf{L} = \begin{pmatrix} 0 & 2\mathbf{K}\mathbf{F}_2 \\ -2\mathbf{K}\mathbf{F}_1 & 0 \end{pmatrix}, \tag{3.13}$$

where $\mathbf{K} = \operatorname{diag}(1/2, 1, 1)$ is a diagonal matrix.

The characteristic polynomial of the linearization matrix $\mathbf{L}$ has the form

$$Q(\lambda) = \lambda^2 Q_1(\lambda), \qquad Q_1(\lambda) = \lambda^4 + k_2\lambda^2 + k_0.$$

The coefficients k_2 and k_0 are defined by the formulas

$$k_2 = 8b_1b_2 - 8d_1c_2 + 8c_1c_2 + a_1^2, \qquad k_0 = 4p_{10}p_{20}. \tag{3.14}$$

The following statements hold.

Proposition 1. *If the conditions*

$$\begin{cases} c_1 + d_1 > 0, \\ p_{j0} > 0, \\ p_{j1} < 0, \quad j = 1, 2, \end{cases} \tag{3.15}$$

are satisfied simultaneously, then all eigenvalues of the matrix $\mathbf{S}$ *are positive, except for the simple zero, and the stationary rotation regime* (2.5), (2.6) *is orbitally stable.*

Proposition 2. *If the conditions*

$$\begin{cases} k_0 \geq 0, \\ k_2 \geq 0, \\ \Delta = k_2^2 - 4k_0 \geq 0 \end{cases} \tag{3.16}$$

are satisfied simultaneously, then all eigenvalues of the linearization matrix $\mathbf{L}$ *lie on the imaginary axis and the solution* (2.5) *is linearly stable.*

If at least one of conditions (3.16) *is violated, then the matrix* $\mathbf{L}$ *has at least one eigenvalue in the right half-plane and the invariant set formed by all orbits of the continuous family* (2.5) *is unstable* (*see* [10, Sect. 3.3] *for details*).

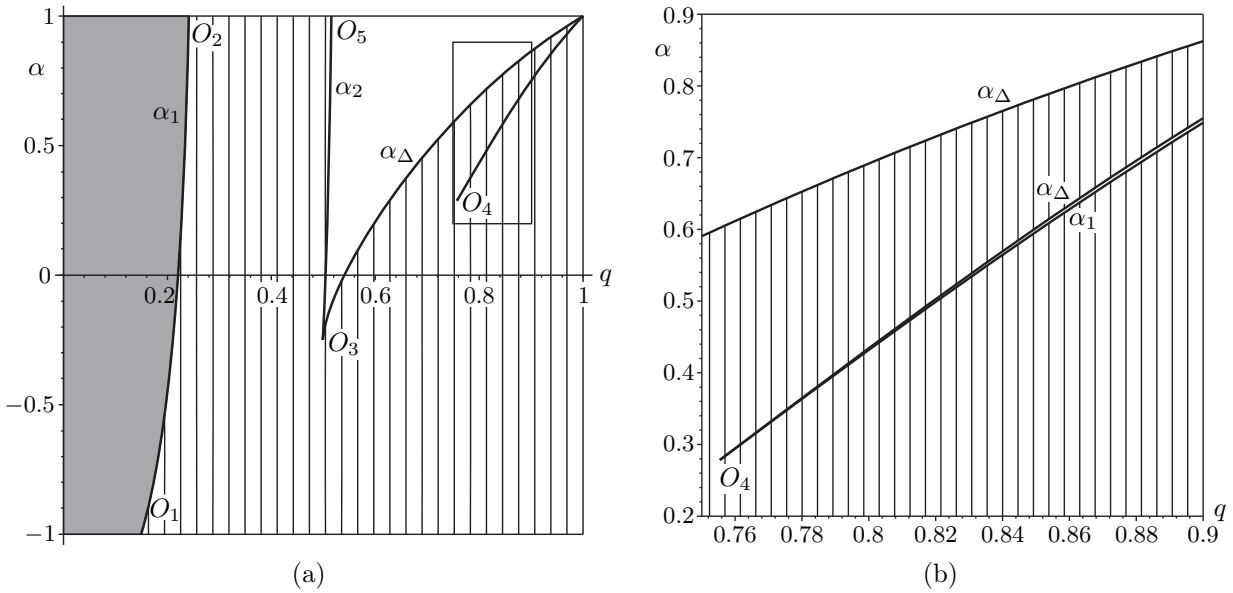

Fig. 1. Stability diagram of the stationary solution (2.5) for zero circulation around the cylinder ($\gamma = 2$) in the case of $n = 2$ in the plane of the parameters $q = R^2/R_0^2$ and $\alpha = (1-a)/(1+a)$: (a) in the region $0 < q < 1$, $-1 < \alpha < 1$ and (b) in the region $0.75 < q < 0.9$, $0.2 < \alpha < 0.9$. Orbital stability (in nonlinear statement) is shown in gray, instability is shown by hatching, and stability in the linear approximation (nonlinear analysis is required) is shown in white.

Figure 1 demonstrates the stability diagram of the stationary solution (2.5) in the parameter plane (q, α), where $0 < q < 1$ and

$$\alpha = \frac{1-a}{1+a}, \qquad -1 < \alpha < 1. \tag{3.17}$$

The gray region is the one in which conditions (3.15) are satisfied, the vertical hatching shows the region where conditions (3.16) are violated, and the white region is the one in which conditions (3.16) are satisfied but conditions (3.15) are not.

The curves $\alpha_j(q)$, $j = 1, 2$, are defined by the conditions $p_{j0}(q, \alpha) = 0$, where the functions p_{j0} are defined by formulas (3.12). The equations of these curves are

$$\alpha_1(q) = -\frac{10q^7 + 10q^6 + 41q^5 - 113q^4 - 48q^3 + 60q^2 + 13q - 5}{10q^7 + 10q^6 - 49q^5 + 17q^4 + 20q^3 + 24q^2 + 3q - 3}, \qquad q \in (0.1485, 0.2406) \cup [0.7561, 1),$$

$$\alpha_2(q) = -\frac{10q^4 + 20q^3 + 15q^2 - 4q - 5}{10q^4 + 20q^3 + 5q^2 - 4q - 3}, \qquad q \in [0.4986, 0.5140).$$

The curve $\alpha_\Delta(q)$ is defined by the equality $\Delta(q, \alpha) = 0$, with Δ defined by (3.16) and (3.14).

The points of intersection of the neutral curves have the following coordinates: $O_1(0.1485, -1)$, $O_2(0.2406, 1)$, $O_3(0.4986, -0.2574)$, $O_4(0.7561, 0.2796)$, and $O_5(0.5140, 1)$.

Nonlinear analysis is needed in two white regions and on their boundaries, as well as on the segment O_1O_2 of the curve α_1. The boundary of the first region is formed by the segment O_3O_5 of the curve α_2 and the branch of the curve α_Δ for $q \in [0.4986, 1)$. The second region is a narrow strip between the segments of the curves α_Δ and α_1 from the point O_4 to the point $(1, 1)$ (see Fig. 1b).

The curves α_2 and α_Δ are tangent to each other at the point O_3, and the curves α_1 and α_Δ, at the point O_4. At both these points, the equality $k_2(q, \alpha) = 0$ holds, where the function k_2 is defined by formula (3.14).

The limit case of $a \to 0$ corresponds to $\alpha \to 1$. As $\alpha \to -1$ ($a \to \infty$), we arrive at the stability problem for Thomson's n-gon in the case of a fixed cylinder for $n = 2$. The point O_1 separates the stability region from the instability region for $\alpha = -1$, which is consistent with the results of Havelock [5]. A nonlinear stability analysis at this point was carried out in [12].

4. CONCLUSIONS

The stability of various configurations of point vortices has been analyzed for many vortex dynamics models (see, for example, the surveys [6–8, 11, 14, 19, 20]).

In the present paper, we have considered the problem of stability of a system of two identical point vortices and a circular cylinder located between them. We assumed that the circulation around the cylinder is zero. To solve this stability problem, one can apply various theories and methods, which have been analyzed in detail in [15, Sect. 2.1], [16, Sect. II.D], and [10, Sect. 3.2]. We mean the stability theory of stationary solutions and invariant sets for dynamical systems with symmetries, the stability theory of stationary regimes and invariant sets for systems with several integrals, the stability theory of a continuous family of cycles of a dynamical system, topological methods of analysis, and so on. In the present work, we have reduced the equations of motion (2.2) and (2.3) by changes of variables to a Hamiltonian system in which a continuous family of equilibria corresponds to the stationary motion (2.5). Orbital stability means the stability of this family (see [10, Sect. 3] for details).

When $a \to \infty$, we arrive at the classical problem of a fixed cylinder and at the well-known results [5, 12] (see Fig. 1 for $\alpha \to -1$). It is somewhat unexpected that the parameter a can lead to additional stability: the region of orbital stability (shown in gray) expands, and additional regions of linear stability (shown in white) arise.

Possible directions of development of the present study involve both the nonlinear analysis of the stability problem in the white region in Fig. 1 and the linear analysis in the case of an arbitrary circulation around the cylinder and an arbitrary number of vortices. Of special interest is the case of zero total intensity, $\gamma = -n$ (see [21]).

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