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Numerical Model of the Baroclinic Instability of Axially Symmetric Eddies in a Two-Layer Ocean

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A numerical quasigeostrophic model is proposed for the evolution of eddy disturbances in a two-layer ocean based on the use of the method of contour dynamics. A detailed study is carried out for the case of an axially symmetric eddy on the Ω-plane. The numerical experiment supports the conclusions of linear stability analysis, and makes it possible to follow the linear stage in the evolution of the system, which breaks down into two eddies with inclined axes diverging to opposite sides.

Introduction. The method of contour dynamics (MCD) [1] which originated as a generalization of the "water bag" model developed for plasma theory [2], in recent years has been applied in increasingly new ways to various oceanological problems [3-6]. A generalization of MCD to the case of a quasi-geostrophic model for an ocean continuously stratified along the vertical has been given in [6]. A problem arises here which is related to a singularity in the tangential velocity component on the contours that separate regions with constant values for the potential eddy. This causes the computational algorithm that has been proposed to be rather sensitive to errors in the approximation. The singularity can be eliminated by changing from continuous stratification while at the same time preserving the important physical mechanisms of the system. As Pedlosky [7] has noted, "baroclinic effects can often be studied with remarkable simplicity in multilayered models."

In this paper MCD is applied to a quasigeostrophic model of a two-layer ocean with a rigid cap at the surface and a rough bottom, taking the β-effect into account. As an example we consider the problem of a baroclinic instability for an axially symmetric eddy on the Ω-plane with a horizontal bottom, where the motion is initiated by a piecewise-constant distribution for the potential vorticity in the layers. The conclusions of the linear stability analysis concerning the predominant azimuthal modes for the disturbances are confirmed by the numerical experiment, which also makes it possible to follow the nonlinear stage in the development of eddy instability, as a result of which the latter breaks up into two eddies with inclined axes, diverging to opposite sides from the common center of the system.

The two-layer MCD model. We consider a two-layer unbounded ocean on the beta plane with a rigid cap at the surface and a rough bottom, whose relative elevation is on the order of the Rossby number $\varepsilon = U^*/\Omega^* L^*$, where the asterisks denote characteristic values for the velocity, Coriolis parameter and the horizontal dimension. In dimensionless variables and in the quasigeostrophic approximation the conservation equations for the potential eddies have the form [7]

$$d_i \Pi_i / dt = 0, \quad i = 1, 2, \tag{1}$$

where the subscripts 1 and 2 pertain to the upper and lower layers and $d_i/dt = \partial/\partial t + u_i \partial/\partial x + v_i \partial/\partial y$. The variable parts of the potential eddies are equal to $\Pi_1 = \Delta p_1 + by + \zeta/d$ and $\Pi_2 = \Delta p_2 + by + (\sigma h + \zeta)/(1-d)$. Here $h(x, y)$ and $\zeta(x, y, t)$ are the elevations for bottom relief and the interface which correspond to the scales h^* and $\varepsilon(H_1 + H_2)$, and the following dimensionless parameters are also introduced: the relative thickness of the upper layers is $d = H_1/(H_1+H_2)$, the planetary one is $b = \beta L^{*2}/U^*$, and the topographic one is $\sigma = h^*/\varepsilon(H_1+H_2)$. The pressures in the layers are related by the dynamic equation

$$\zeta = F(p_2 - p_1), \tag{2}$$

where the Froude number is expressed in terms of the inner radius of curvature $R_d = [g(\rho_2-\rho_1)H_1H_2/\rho_0\Omega^{*2}(H_1+H_2)]^{1/2}$ according to the equation $F = k^2(1-d)d$, $k = L^*/R_d$.

Introducing the barotropic stream function $\psi = dp_1 + (1-d)p_2$, from (2) we have

$$p_1 = \psi - (1-d)(\zeta/F), \quad p_2 = \psi + d(\zeta/F). \tag{3}$$

In order to apply the MCD, it is sufficient to assume that the Π_i are piecewise-constant at the

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initial point in time; because of the conservation laws (1) this property also becomes valid at any succeeding time. For simplicity we assume that the Π_i are different from zero in the singly connected regions S_i with boundaries C_i.

We use ψ_0 and ζ_0 to denote the general solutions of the equations $\Delta\psi=-\sigma h-by$, $\Delta\zeta-k^2\zeta=-\sigma Fh/(1-d)$. It is easily shown that

$$p_1(x,y,t)=p_{10}+\Pi_1\iint_{S_1} G(R)\,d\xi d\eta-(1-d)p,$$
$$p_2(x,y,t)=p_{20}+\Pi_2\iint_{S_2} G(R)\,d\xi d\eta+dp,$$
$$p(x,y,t)=\Pi_2\iint_{S_2}\hat{G}(R)\,d\xi d\eta-\Pi_1\iint_{S_1}\hat{G}(R)\,d\xi d\eta,$$

where the pressures determining the "external" field in the layers $p_{i0}(x,y,t)$ are expressed in terms of ψ_0 and ζ_0 according to Eqs. (3), and $\hat{G}=G_0-G$ is the difference between the Green's functions for the Helmholtz and Laplace operators: $G_0(R)=-(1/2\pi)K_0(kR)$, $G(R)=(1/2\pi)\ln R$, $R=[(x-\xi)^2+(y+\eta)^2]$. Applying the equations $u_i=-p_{iy}$ and $v_i=p_{ix}$, and transforming the double integrals to contours, we obtain equations for the velocity vectors in the layers

$$\mathbf{V}_1=\mathbf{V}_{10}+\Pi_1\mathbf{V}^{(c_1)}-(1-d)\mathbf{V},$$
$$\mathbf{V}_2=\mathbf{V}_{20}+\Pi_2\mathbf{V}^{(c_2)}+d\mathbf{V},\quad \mathbf{V}=\Pi_2\hat{\mathbf{V}}^{(c_2)}-\Pi_1\hat{\mathbf{V}}^{(c_1)},$$

where we have used the notation

$$\mathbf{V}^{(c)}=-\oint_c G(R)\,d\rho,\ \hat{\mathbf{V}}=-\oint_c\hat{G}(R)\,d\rho,\ d\rho=(d\xi,d\eta).\quad (4)$$

The second contour integral in (4) does not have singularities in the function under the integral and thus it is easily realized numerically. The first integral determines the velocity field in the barotropic problem and also can be made regular [3].

The motion of the reference points on the motion C_i determines the equations

$$dr_j^{(i)}/dt=\mathbf{V}_j^{(i)},\ i=1,2,\ j=\overline{1,N_i},\quad (5)$$

where $\mathbf{r}_j^{(i)}$ is the radius vector of the j-th fixed fluid particle on the i-th contour. Further application of the MCD is standard and uses the computational procedures described in [3-5].

Linear stability analysis. The sufficient conditions that permit the application of the MCD in two-layer quasigeostrophic models have been formulated above. We give the example of a purely zonal "external" field when $h=h(x,y)$. In this case

$$\psi_0=-U(t)y-by^3/6-\sigma\int_0^y h(\eta)(y-\eta)\,d\eta,$$

$$\zeta_0=c_1(t)e^{-ky}+c_2(t)e^{ky}+\sigma F/2k(1-d)\times$$
$$\times\left[\int_{-\infty}^{y}e^{-k(y-\eta)}h(\eta)\,d\eta+\int_{y}^{\infty}e^{-k(\eta-y)}h(\eta)\,d\eta\right].$$

In this way the dynamics of the eddy regimes S_i can be studied against the background of a given effect (external flow, bottom relief of beta-effect). When $b=\sigma=0$ and $C_1(t)=C_2(t)\equiv0$ we obtain a barotropic current with velocity $U(t)$ along the x-axis. Next we consider the special case when the "external" field is absent ($\psi_0=\zeta_0=0$), which is possible only for a uniform bottom and when the beta-effect disappears ($h\equiv0$, $b=0$). Every auxiliary symmetric solution of the problem with an arbitrary radial distribution for the potential eddies $\Pi_i(r)$ is steady-state. We shall formulate the spectral problem determining the stability parameters for these states, where in contrast to the traditional approaches in [8] we will use essentially the integral form for the solution.

In the development of the instability let the lines for constant potential eddies in polar coordinates be defined by the equations $r_i=f_i(\theta,t;\alpha)$ for $i=1$ and 2 where the meaning of the parameter α is clear from the condition $\alpha=f_i(\theta,0;\alpha)$. Because of the conservation laws (1) these lines match the fluid contours so that their differential equations of motion have the form

$$f_if_{it}+V_i^{(\theta)}f_{i\theta}-V_i^{(r)}f_i=0,\ i=1,2,\quad (6)$$

where $V^{(r)}$ and $V^{(\theta)}$ are the radial and azimuthal velocity projections. We use the method of small perturbations, setting $f_i(\theta,t;\alpha)=\alpha+\varepsilon_i(\theta,t;\alpha)$, where $|\varepsilon_i|\ll\alpha$. Linearizing the equations (6) we obtain

$$\varepsilon_{it}+(1/\alpha)\varepsilon_{i\theta}\bar{V}_i^{(\theta)}-\tilde{V}_i^{(r)}=0,\ i=1,2,\quad (7)$$

where the symbols (-) and (~) pertain to the unperturbed ($\varepsilon_i\equiv0$) and perturbed (linearly related to ε_i) sections of the velocity field which are calculated by the equations following from (3)

$$\bar{V}_1^{(\theta)}=\partial\bar{\psi}/\partial r-[(1-d)/F]\,\partial\bar{\zeta}/\partial r,\ \tilde{V}_1^{(r)}=$$
$$-(1/r)\,\partial\tilde{\psi}/\partial\theta+[(1-d)/Fr]\,\partial\tilde{\zeta}/\partial\theta,$$
$$\bar{V}_2^{(\theta)}=\partial\bar{\psi}/\partial r+d\,\partial\bar{\zeta}/\partial r,\ \tilde{V}_2^{(r)}=$$
$$-(1/r)\,\partial\tilde{\psi}/\partial\theta-(d/Fr)\,\partial\tilde{\zeta}/\partial\theta,\quad (8)$$

where

$$\bar{\psi}+\tilde{\psi}=\iint[d\Pi_1+(1-d)\Pi_2]\,G(R)\,d\xi d\eta,\quad (9)$$

$$\bar{\zeta}+\tilde{\zeta}=F\iint(\Pi_2-\Pi_1)\,G_0(R)\,d\xi d\eta.\quad (10)$$

For the individual perturbation modes we use $\varepsilon_i(\theta,t;\alpha)=A_i(\alpha)\exp\cdot[im(\theta-\gamma t)]$, $m\geqslant1$. Transforming (9) and (10) to the new integration variables

(φ, β) using $\xi = f_i(\varphi, t; \beta)\cos\varphi$, and $\eta = f_i(\varphi, t; \beta)\sin\varphi$ with $J = f_i f_{i\beta}$ the Jacobian of the transformation, and taking into account that to within the accuracy of linear terms relative to ε_i, $GJ = \beta G + \partial[G\beta\varepsilon_i]/\partial\beta$, after integration by parts with respect to β and performing quadratures with respect to the angle φ, we obtain

$$\bar{\psi} = \int_0^\infty [d\Pi_1(\beta) + (1-d)\Pi_2(\beta)]\beta T^{(0)}(r,\beta)\,d\beta, \quad (11)$$

$$\bar{\zeta} = F\int_0^\infty [\Pi_2(\beta) - \Pi_1(\beta)]\beta T_0^{(0)}(r,\beta)\,d\beta, \quad (12)$$

$$\tilde{\psi} = -e^{im(\theta-\gamma t)}\int_0^\infty [d\Pi_1'(\beta)A_1(\beta) + (1-d)\Pi_2'(\beta)A_2(\beta)]\beta T^{(m)}(r,\beta)\,d\beta, \quad (13)$$

$$\tilde{\zeta} = -e^{im(\theta-\gamma t)}F\int_0^\infty [\Pi_2'(\beta)A_2(\beta) - \Pi_1'(\beta)A_1(\beta)]\beta T^{(m)}(r,\beta)\,d\beta, \quad (14)$$

where

$$T^{(m)}(r,\beta) = \int_0^{2\pi} G(R)e^{im\varphi}d\varphi, \quad T_0^{(m)}(r,\beta) = \int_0^{2\pi} G_0(R)e^{im\varphi}d\varphi, \quad (15)$$

with $R = [r^2+\beta^2-2r\beta\cos\varphi]^{1/2}$.

The integrals in (15) are easily calculated using the "multiplication theorems" in [9] for cylindrical functions:

$$T^{(m)}(\alpha,\beta) = -\frac{1}{2m}\begin{cases}(\alpha/\beta)^m,\\(\beta/\alpha)^m,\end{cases}$$

$$T_0^{(m)}(\alpha,\beta) = -\begin{cases}I_m(k\alpha)K_m(k\beta), & \alpha<\beta,\\ I_m(k\beta)K_m(k\alpha), & \alpha>\beta.\end{cases}$$

Substituting (8) in (7) and taking (11)-(14) into account, we have finally a homogeneous system of integral equations

$$[-\gamma+P(\alpha)-(1-d)Q(\alpha)]A_1(\alpha)-M_m(\alpha)+(1-d)N_m(\alpha)=0, \quad (16)$$

$$[-\gamma+P(\alpha)+dQ(\alpha)]A_2(\alpha)-M_m(\alpha)-dN_m(\alpha)=0, \quad (17)$$

$$P(\alpha) = (1/\alpha^2)\int_0^\alpha [d\Pi_1(\beta)+(1-d)\Pi_2(\beta)]\beta d\beta,$$

$$Q(\alpha) = (k/\alpha)\left\{K_1(k\alpha)\int_0^\alpha [\Pi_2(\beta)-\Pi_1(\beta)]\beta I_0(k\beta)\,d\beta - I_1(k\alpha)\int_\alpha^\infty [\Pi_2(\beta)-\Pi_1(\beta)]\beta K_0(k\beta)\,d\beta\right\},$$

$$M_m(\alpha) = (1/\alpha)\int_0^\infty [d\Pi_1'(\beta)A_1(\beta)+(1-d)\Pi_2'(\beta)A_2(\beta)]\beta T^{(m)}(\alpha,\beta)\,d\beta,$$

$$N_m(\alpha) = (1/\alpha)\int_0^\infty [\Pi_2'(\beta)A_2(\beta)-\Pi_1'(\beta)A_1(\beta)]\beta T_0^{(m)}(\alpha,\beta)\,d\beta.$$

The spectral stability problem (16) and (17) formulated is suitable for arbitrary $\Pi_1(r)$ and $\Pi_2(r)$, but is especially convenient for piecewise-constant distributions for the potential vorticity that are characteristic of MCD. We consider the simple case when

$$\Pi_1(r) = \Pi_1 H(a_1-r), \quad \Pi_2(r) = \Pi_2 H(a_2-r), \quad (18)$$

where $H(r)$ is the unitary Heaviside function. Writing Eqs. (16) and (17) with $\alpha = a_1$ and $\alpha = a_2$ we obtain a homogeneous system of four linear algebraic equations relative to $A_i(a_j)$, $i, j = 1, 2$. Setting the determinant equal to zero we find the dispersion equation

$$[(\gamma-2X)(\gamma-2Y)-Z][\gamma-S(a_2)][\gamma-T(a_1)]=0, \quad (19)$$

$$X = (1/2)\{S(a_1)+\Pi_1[(1-d)T_0^{(m)}(a_1,a_1)+dT^{(m)}(a_1,a_1)]\},$$

$$Y = (1/2)\{T(a_2)+\Pi_2[(1-d)T^{(m)}(a_2,a_2)+dT_0^{(m)}(a_2,a_2)]\},$$

$$z = d(1-d)\Pi_1\Pi_2[T^{(m)}(a_1,a_2)-T_0^{(m)}(a_1,a_2)]^2,$$

$$S(a) = P(a)-(1-d)Q(a), \quad T(a) = P(a)+dQ(a).$$

The roots of the quadratic trinomial in (19) have the form $\gamma = X+Y\pm\sqrt{D}$ and $D = (X-Y)^2+Z$, so therefore the instability condition is expressed by the inequality $D<0$ from which, in particular, it follows that this requires $\Pi_1\Pi_2 < 0$.

We have a four-parameter problem including four determining ratios, namely for the depths d, the characteristic horizontal dimension to the radius of deformation k, the potential eddies Π_1/Π_2, and the radii a_1/a_2. To simplify the analysis we set $a_1 = a_2 = 1$ and $d\Pi_1+(1-d)\Pi_2=0$, which excludes the barotropic mode of motion at the initial time. For the following it is convenient to use the representation $\Pi_1 = (1-d)\mu$, and $\Pi_2 = -d\mu$ with parameter μ. Then the problem reduces to an analysis of the dispersion relation $\gamma_m = (\mu/2)\{(1-2d)[L_1(k)-L_m(k)] \pm \sqrt{\Phi_m(k,d)}\}$, where $L_m = I_m(k)K_m(k)$ and

$$\Phi_m(k,d) = [L_1(k)-L_m(k)][L_1(k)-(1-2d)^2L_m(k)-(2/m)d(1-d)]. \quad (20)$$

The function $L_m(k)$ decreases monotonically with increasing argument k and parameter m, with $L_m(k)\to 1/2m$ as $k\to 0$ and $L_m(k)\sim 1/2k$ as $k\to\infty$. It is clear that $\gamma_1 = 0$. For $m \geq 2$ we obtain the neutral stability curves equating the second factor in (20) to zero, from which $d = (1/2)\{1 \pm [(1/2m-L_1)(1/2m-L_m)]^{1/2}\}$ follows. The neutral stability curves constructed using this equation are shown in Fig. 1, where the m-th mode becomes unstable above the m-th curve. We denote the root of the equation $L_1(k) = 1/2m$ by k_m; asymptotically $k_m \sim m$. For $k < k_2 = 1.7$ the condition is stable for all d.

Numerical experiment. The predictions of the linear theory described above were verified by means of numerical experiments based on MCD. One of the examples is shown in Fig. 2, where the upper (I) and lower (II) series correspond to the upper and lower layers, and the times are indicated below. The numerical value for μ was determined

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Numerical experiment. The predictions of the linear theory described above were verified by means of numerical experiments based on MCD. One of the examples is shown in Fig. 2, where the upper (I) and lower (II) series correspond to the upper and lower layers, and the times are indicated below. The numerical value for μ was determined from the condition $\Pi_1 L_1(k)=1$ for which the azimuthal velocity on the upper contour is equal to unity at $t = 0$. At the initial time 40 uniformly distributed reference points were used to approximate the circular contours. The equations of motion (5) were integrated by the Euler method with $\Delta t = 0.1$ for the time step calculation. After each step the reference points were recalculated in proportion to the increasing length for each contour. In the experiment we took $d = 0.2$ and $k = 2.6$ (Fig. 1, point), which corresponds to instability for the mode with $m = 2$. The errors in the approximation scheme and rounding off in the computer were a continuous source of perturbations. Under the influence of this "noise" the two-layer eddy maintained its initial axially symmetric form for a rather long time, but by $t \gtrsim 60$ excitation of the mode with $m = 2$ already clearly began to be evident, which then rapidly went over to a nonlinear instability situation and led to the breakup of the initial configuration into two dual-layer eddies. Because of the horizontal shift in the center of gravity for the potential vorticity in the layers, each of the newly formed dual-layer eddies behaves as a pair of singular geostrophic eddies [10], both of them diverging to opposite sides of the common center of the system. Similar results had been obtained earlier by Ikeda [11] who integrated the equations for the dual-layer model by a grid method with periodic boundary conditions, assigning the initial disturbances for the original axially symmetric eddy according to the theoretical unstable mode. He considered a fixed ratio of $d = 0.3$ for the thicknesses of the layers, varying the Froude number (the parameter k) and the ratio of the amplitudes for the velocities in the layers (the analog of our Π_1 / Π_2). In our problem the direction of dispersal for the pairs is set randomly (compare [5])). This is borne out by numerical experiments in which various small disturbances were set up for the axially symmetric contours C_i at the initial time. When the beta-effect is taken into account the direction becomes zonal, which apparently should lead to the eastward propagation for the resulting modon as occurred in the experiments in [12,13].

Conclusion. We present a brief summary of the results. For instability in an axially symmetric two-layer eddy with constant potential vorticity in the layers Π_1 and Π_2 it is necessary to satisfy the condition $\Pi_1\Pi_2<0$. This requirement is satisfied by eddies with a null barotropic mode for which $d\Pi_1+(1-d)\Pi_2=0$ and the bearers of constant vorticity agree ($a_1 = a_2$). The boundary for stability is defined by the equation $\Phi_2(k, d)=0$ in the plane of the parameters d and k. The eddies are known to be stable when $k < k_2 = 1.7$, where $L_1(k_1)={}^1/_4$, the minimum stability occurring at $d = \frac{1}{2}$. For any k and d from the region of definition for these parameters satisfying the condition $\Phi_2(k, d)<0$, there is always a finite number of perturbation modes contained within the interval $2\leqslant m\leqslant M$, where $\Phi_M(k, d)\geqslant 0$. The number of unstable modes increases with increasing k and decreases with increasing $|d-\frac{1}{2}|$.

The first unstable mode is the one with $m = 2$. If this is a unitary unstable mode, during evolution the eddy breaks down into two individual eddies with inclined axes diverging in opposite directions similar to vortex pairs. We note that the eddies with opposite rotation in different layers with an inclined axis have actually been observed in the ocean [14].

An advantage of the MCD models, besides reducing the dimensionality of the problem, is the ease of interpretation and the clear manner for representing the results. As a rule, analysis of the stability of stationary states using these models turns out to be clearer and more straightforward. This is supported by comparison with [11] where similar conclusions were obtained by a more complex and involved method. In our opinion this shows the promising nature for applying MCD to model a broad class of interesting and important oceanological problems.

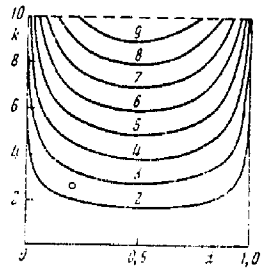

Fig. 1. Neutral stability curves for individual perturbation modes; the point corresponds to parameters $d = 0.2$ and $k = 2.6$ in the numerical experiment.

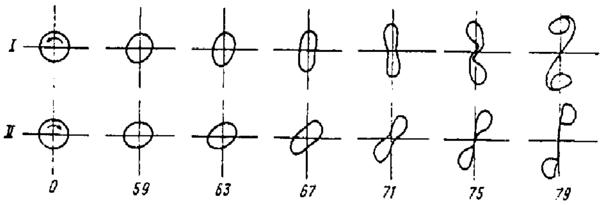

Fig. 2. Decay of an unstable two-layer eddy, numerical experiment; dimensionless times are shown beneath the corresponding boundaries for regions of constant vorticity in the upper (I) and lower (II) layers.

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